CALCULUS.

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101. **Proposed by WILLIAM FRED FLEMING, Denison, Tex.**

A 24-inch joint of 6-inch stove pipe is compressed at one end to make it fit over an elliptical opening in a stove (for the escape of smoke). The ellipse has a major axis of 8 inches. What reduction is there in the solid contents of the stove pipe, assuming that its compressed shape may be generated by a 6-inch circle which passes uniformly from one end to the other and perpendicular to the axis of the pipe?

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

The perimeter of an ellipse semi-major axis a is

$$4a\int_0^{\frac{1}{2}\pi}\sqrt{1-e^2\sin^2\theta}\,d\theta=s.$$

In this problem $a=4$ and $s=6\pi$, the circumference of the pipe.

$$\therefore\ 6\pi=16\int_0^{\frac{1}{2}\pi}(1-\tfrac{1}{2}e^2\sin^2\theta-\tfrac{1}{8}e^4\sin^4\theta-\tfrac{1}{16}e^6\sin^6\theta-\text{etc.})d\theta$$

$$=8\pi(1-\tfrac{1}{4}e^2-\tfrac{3}{64}e^4-\tfrac{5}{256}e^6-\tfrac{175}{16384}e^8-\tfrac{441}{65536}e^{10}-\text{etc.})$$

Let $e=u$. Then $u+\frac{3}{16}u^2+\frac{5}{64}u^3+\frac{175}{4096}u^4+\frac{441}{16384}u^5+\ldots=1$.
By reversion of series $u=e^2=.8013$.
$\therefore\ b=a\sqrt{(1-e^2)}=4\sqrt{(.1987)}=1.783$ inches.

Let the pipe be compressed uniformly for the entire length. Also let πxy be the area of any elliptical section, and z the distance of this section from the circular end of the pipe.

$$\text{Then } x=\frac{72+z}{24},\ y=\frac{72-1.217z}{24}.$$

$$V=\pi\int_0^{24}xy\,dz=\frac{\pi}{576}\int_0^{24}(5184-15.624z-1.217z^2)dz.$$

$\therefore\ V=624.4568$ cubic inches.
$V_1=$volume before compression.
$\therefore\ V_1=9\pi\times24=678.5856$ cubic inches.
$\therefore\ V_1-V=54.1288$ cubic inches reduction in volume.

II. **Solution by H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.**

Denote the length of the pipe ($=24$ inches) by h, the semi-major axis ($=4$) of the ellipse by a, the semi-minor axis by b, the eccentricity by e, the radius of the circle ($=3$) by r. Then equating semi-perimeters:

$$\pi r = \pi a\left(1 - \frac{e^2}{2^2} - \frac{1^2.3.e^4}{2^2.4^2} - \frac{1^2.3^2.5.e^6}{2^2.4^2.6^2} - \right).$$

Solving by trial, $e=.895$, and $b=1.784$.

At a distance of z from the circular end of the pipe, the section is an ellipse, the semi-axes of which are found to be

$$\frac{hr-rz+az}{h} \text{ and } \frac{hr-rz+bz}{h}.$$

Hence the volume is

$$\int_0^h \pi\left(\frac{hr-rz+az}{h}\right)\left(\frac{hr-rz+bz}{h}\right)dz = \frac{\pi h}{6}(ar+br+2r^2+2ab) = 198.5\pi.$$

Hence the loss $=216\pi-198.5\pi=17.5\pi=55$ cubic inches.

MISCELLANEOUS.

82. Proposed by A. H. BELL, Hillsboro, Ill.

Four spheres of equal radii $=r=5$, are in contact, and form a triangular pyramid. How large is the sphere that can be placed in the middle and be in contact with the four spheres.

Solution by J. W. YOUNG, Fellow in Mathematics, Cornell University, Ithaca, N. Y., and J. SCHEFFER, A. M., Hagerstown, Md.

Let A, B, C, B' (Fig. 1) be the centers of the four spheres. They evidently form the corners of a regular tetrahedron. Fig. 2 is a picture of a plane section of the pyramid of spheres, passed through the points $AB'L$, where L is the point of tangency of the two spheres (C, B).

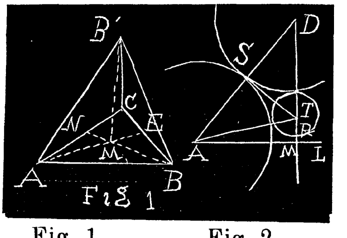

Fig. 1. Fig. 2.

From Fig. 1,

$$\left.\begin{array}{l} AN/AM=\sin 60^\circ \\ AN/AB'=\cos 60^\circ \end{array}\right\}.$$

$\therefore AB'/AM=\tan 60^\circ=\sqrt{3}=\sec DAM$.

In Fig. 2, then, $\angle DAM$ is $\sec^{-1}\sqrt{3}$. It is clear that the required small sphere must have its center on DM and must touch both spheres (A, D). Let $\angle ADM=\theta$.

Then $\sin\theta=1/\sqrt{3}$, $\cos\theta=\sqrt{\tfrac{2}{3}}$, $DT/r=\sqrt{\tfrac{3}{2}}$.

$\therefore DT=(r/2)\sqrt{6}$.

$\therefore RT=DT-r=(r/2)(\sqrt{6}-2)=$radius of small sphere.

$r=5$ gives $RT=1.1238$.

85. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.